

POSITIVE EXPRESSIONS FOR SKEW DIVIDED DIFFERENCE OPERATORS

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ABSTRACT. For permutations $v, w \in \mathfrak{S}_n$, Macdonald defines the skew divided difference operators $\partial_{w/v}$ as the unique linear operators satisfying $\partial_w(PQ) = \sum_v v(\partial_{w/v}P) \cdot \partial_vQ$ for all polynomials P and Q . We prove that $\partial_{w/v}$ has a positive expression in terms of divided difference operators ∂_{ij} for $i < j$. In fact, we prove that the analogous result holds in the Fomin-Kirillov algebra \mathcal{E}_n , which settles a conjecture of Kirillov.

1. INTRODUCTION

The divided difference operators ∂_{ij} acting on $\mathbf{C}[x_1, \dots, x_n]$ are vital in the study of Schubert calculus. Their main purpose is to define Schubert polynomials, which serve as polynomial representatives of Schubert classes in the cohomology ring of the flag variety. Finding a combinatorial formula for the structure constants c_{uv}^w of Schubert polynomials is a long outstanding problem in algebraic combinatorics.

Macdonald [11] defined for any permutations v and w a skew divided difference operator $\partial_{w/v}$ such that applying $\partial_{w/v}$ to the Schubert polynomial of a permutation u with $\ell(u) + \ell(v) = \ell(w)$ gives the structure constant c_{uv}^w . In [9], Kirillov conjectures that $\partial_{w/v}$ can be written as a polynomial in ∂_{ij} for $i < j$ with positive coefficients. The main result of this paper is to prove this conjecture.

In fact, Kirillov conjectures a slightly more general result. The divided difference operators give a representation of a larger algebra \mathcal{E}_n introduced by Fomin and Kirillov [5]. Kirillov then conjectures that the element of the Fomin-Kirillov algebra corresponding to $\partial_{w/v}$ has a positive expression in terms of generators $x_{ij} \in \mathcal{E}_n$ for $i < j$. This form of positivity in \mathcal{E}_n is notable due to the nonnegativity conjecture in [5], which states that certain elements of \mathcal{E}_n (namely evaluations of Schubert polynomials at Dunkl elements) have such a positive expression. A proof of this nonnegativity conjecture together with an explicit positive expression for such elements would immediately give a combinatorial formula for the structure constants c_{uv}^w .

The Fomin-Kirillov algebra also has the structure of a braided Hopf algebra as noted in [6, 12]. This added structure (which does not exist in full for the quotient algebra generated by the divided difference operators) will be key in proving our main theorem.

We begin with some preliminaries about the symmetric group, divided difference operators, and the braided Hopf algebra structure of the Fomin-Kirillov algebra \mathcal{E}_n in Section 2. We then prove the main result in Theorem 3.5 of Section 3, giving a positive explicit formula for $\partial_{w/v}$ in Corollary 3.6 and a positive recursive formula in Corollary 3.8.

2. PRELIMINARIES

In this section, we give some notation and background, and we prove some basic facts about divided difference operators and the Fomin-Kirillov algebra. For more information, see, for instance, [8, 9, 11].

2.1. Symmetric group. Let \mathfrak{S}_n be the symmetric group on n letters. We will write s_{ij} for the transposition switching i and j , and we will abbreviate the simple transposition $s_{i,i+1}$ by s_i .

Given an element $w \in \mathfrak{S}_n$, a *reduced expression* for w is a factorization of w into simple transpositions $s_{i_1} \cdots s_{i_\ell}$ of minimum length $\ell = \ell(w)$. Any two reduced expressions for w can be obtained from one another by commuting s_i and s_j when $|i - j| > 1$ or by applying *braid moves* replacing $s_i s_j s_i$ with $s_j s_i s_j$ when $|i - j| = 1$.

We say $w = u \cdot v$ is a *reduced factorization* if $\ell(w) = \ell(u) + \ell(v)$. We also denote the longest element of \mathfrak{S}_n by w_0 , so $\ell(w_0) = \binom{n}{2}$.

If some (equivalently, any) reduced expression for w contains a subsequence that is a reduced expression for v , we say that $v < w$ in *Bruhat order*. Equivalently, v is covered by w in Bruhat order, written $v \triangleleft w$, if $\ell(v) = \ell(w) - 1$ and $v = w s_{ij}$ for some transposition s_{ij} .

2.2. Divided difference operators. Define the left action of \mathfrak{S}_n on $\mathbf{C}[x_1, \dots, x_n]$ by

$$(wP)(x_1, \dots, x_n) = P(x_{w(1)}, \dots, x_{w(n)}).$$

The *divided difference operator* ∂_{ij} is then defined by

$$\partial_{ij}P = \frac{P - s_{ij}P}{x_i - x_j}$$

for distinct i and j . We abbreviate $\partial_{i,i+1}$ by ∂_i .

The following proposition describes some simple but important properties of the divided difference operators.

Proposition 2.1. *The divided difference operators satisfy, for distinct i, j, k and l :*

- (a) $\partial_{ij} = -\partial_{ji}$;
- (b) $\partial_{ij}^2 = 0$;
- (c) $\partial_{ij}\partial_{kl} = \partial_{kl}\partial_{ij}$;
- (d) $\partial_{ij}\partial_{jk} + \partial_{jk}\partial_{ki} + \partial_{ki}\partial_{ij} = 0$;
- (e) $\partial_{ij}\partial_{jk}\partial_{ij} = \partial_{jk}\partial_{ij}\partial_{jk} = \partial_{ij}\partial_{ik}\partial_{jk} = \partial_{jk}\partial_{ik}\partial_{ij}$;
- (f) $\partial_{ij}(PQ) = \partial_{ij}P \cdot Q + s_{ij}P \cdot \partial_{ij}Q$; and
- (g) $\partial_{ij}w = w\partial_{w^{-1}(i)w^{-1}(j)}$ for all $w \in \mathfrak{S}_n$.

Proof. Straightforward computation. □

Proposition 2.1(e) follows from (a), (b), and (d). The first equality is the *braid relation*, while the last equality is sometimes called the *Yang-Baxter equation*.

The divided difference operators satisfy other relations that are not implied by those in Proposition 2.1, but we will not need them here. However, the only relations between the simple divided difference operators ∂_i are (b), (c), and the braid relation in (e). These relations define the *nil-Coxeter algebra* of \mathfrak{S}_n . Thus $\partial_1, \dots, \partial_{n-1}$ generate a faithful representation of the nil-Coxeter algebra (see [7]). Given a permutation $w \in \mathfrak{S}_n$, let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression, and define $\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$. Then ∂_w does not depend on the choice of reduced

expression for w . For any two permutations $v, w \in \mathfrak{S}_n$, $\partial_v \partial_w = \partial_{vw}$ if $\ell(vw) = \ell(v) + \ell(w)$ and 0 otherwise.

By applying Proposition 2.1(f) repeatedly to $\partial_w(PQ) = \partial_{i_1} \cdots \partial_{i_\ell}(PQ)$, we can express $\partial_w(PQ)$ as

$$\partial_w(PQ) = \sum_{v \in \mathfrak{S}_n} v(\partial_{w/v} P) \cdot \partial_v Q$$

for some linear operators $\partial_{w/v}$. These operators are called the *skew divided difference operators*. The operator $\partial_{w/v}$ reduces the degree of a polynomial by $\ell(w) - \ell(v)$.

Example 2.2. Let $w = s_1 s_2 \in \mathfrak{S}_3$. Then

$$\begin{aligned} \partial_w(PQ) &= \partial_1 \partial_2(PQ) \\ &= \partial_1(\partial_2 P \cdot Q + s_2 P \cdot \partial_2 Q) \\ &= \partial_1 \partial_2 P \cdot Q + s_1 \partial_2 P \cdot \partial_1 Q + \partial_1 s_2 P \cdot \partial_2 Q + s_1 s_2 P \cdot \partial_1 \partial_2 Q \\ &= \partial_1 \partial_2 P \cdot Q + s_1 \partial_2 P \cdot \partial_1 Q + s_2 \partial_{13} P \cdot \partial_2 Q + s_1 s_2 P \cdot \partial_1 \partial_2 Q. \end{aligned}$$

Hence $\partial_{w/id} = \partial_1 \partial_2$, $\partial_{w/s_1} = \partial_2$, $\partial_{w/s_2} = \partial_{13}$, and $\partial_{w/w} = 1$.

Alternatively, $\partial_{w/v}$ can be calculated as follows. Again consider any reduced expression $w = s_{i_1} \dots s_{i_\ell}$. For any subset $J \subseteq \{1, \dots, \ell\}$, let $\varphi_J = \prod_{j=1}^\ell \varphi_j(J)$, where $\varphi_j(J) = s_{i_j}$ if $j \in J$ and ∂_{i_j} if $j \notin J$. Then

$$\partial_{w/v} = v^{-1} \sum_J \varphi_J,$$

where J ranges over all subsets for which the product $\prod_{j \in J} s_{i_j}$ is a reduced expression for v . Note that this immediately implies that $\partial_{w/v} = 0$ unless $v < w$ in Bruhat order.

By using Proposition 2.1(g) to collect the transpositions in φ_J to the left, we will always be able to write $\partial_{w/v}$ as a polynomial in the divided difference operators ∂_{i_j} .

Example 2.3. Let $w = s_2 s_1 s_3 s_2 \in \mathfrak{S}_4$, and let $v = s_2$. Then the only possibilities for J that yield a reduced expression for v are $J = \{1\}$ and $J = \{4\}$. Thus

$$\begin{aligned} \partial_{w/v} &= v^{-1}(s_2 \partial_{12} \partial_{34} \partial_{23} + \partial_{23} \partial_{12} \partial_{34} s_2) \\ &= s_2(s_2 \partial_{12} \partial_{34} \partial_{23} + s_2 \partial_{32} \partial_{13} \partial_{24}) \\ &= \partial_{12} \partial_{34} \partial_{23} - \partial_{23} \partial_{13} \partial_{24}. \end{aligned}$$

Note that while this expression has a negative sign in it, we can use Proposition 2.1(d) to rewrite it as

$$\begin{aligned} \partial_{12} \partial_{34} \partial_{23} - \partial_{23} \partial_{13} \partial_{24} &= \partial_{12}(\partial_{23} \partial_{24} + \partial_{24} \partial_{34}) - (\partial_{12} \partial_{23} - \partial_{13} \partial_{12}) \partial_{24} \\ &= \partial_{12} \partial_{24} \partial_{34} + \partial_{13} \partial_{12} \partial_{24}. \end{aligned}$$

This example shows that naïve evaluation of $\partial_{w/v}$ will generally not give an expression in ∂_{i_j} with $i < j$ that has positive coefficients. The main result of this paper will be to prove the following theorem, which states that such a positive expression always exists. It will follow as an immediate consequence of Theorem 3.5 below.

Theorem 2.4. *For any $v, w \in \mathfrak{S}_n$, the skew divided difference operator $\partial_{w/v}$ can be written as a polynomial with nonnegative coefficients in the operators ∂_{i_j} for $i < j$.*

An explicit expression for $\partial_{w/v}$ follows from Corollary 3.6.

2.3. Fomin-Kirillov algebra. We will not need the full set of relations between the ∂_{ij} to prove Theorem 2.4, only the ones appearing in Proposition 2.1. As such, it will be helpful to work inside the *Fomin-Kirillov algebra* \mathcal{E}_n , which is essentially defined by the relations in Proposition 2.1(a)–(d). (Relation (e) can be derived from these four.) See [3, 5] for more information.

Definition. The Fomin-Kirillov algebra \mathcal{E}_n is the (noncommutative) algebra with generators $x_{ij} = -x_{ji}$ for $1 \leq i < j \leq n$ satisfying the following relations for distinct i, j, k , and l :

- $x_{ij}^2 = 0$;
- $x_{ij}x_{kl} = x_{kl}x_{ij}$; and
- $x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0$.

The divided difference operators therefore give a representation of \mathcal{E}_n , though this representation is not faithful for $n \geq 3$. As with divided difference operators, the subalgebra generated by $x_{12}, x_{23}, \dots, x_{n-1,n}$ is isomorphic to the nil-Coxeter algebra. Given any $w \in \mathfrak{S}_n$ with reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, we will define $x_w = x_{i_1, i_1+1} \cdots x_{i_\ell, i_\ell+1} \in \mathcal{E}_n$. We can similarly define $x_{w/v} \in \mathcal{E}_n$ as before (or one can take Proposition 2.7 below as a definition).

One important property of \mathcal{E}_n is that it has a large amount of structure: in particular, it is a *braided Hopf algebra*. (This is not the case for the algebra generated by divided difference operators, which is a quotient of \mathcal{E}_n .) We review some properties of this structure below. For more details about this structure and braided Hopf algebras in general, see [1, 3, 6, 12].

2.3.1. Grading and braiding. In addition to the usual degree grading, the Fomin-Kirillov algebra has an \mathfrak{S}_n -grading: define the \mathfrak{S}_n -degree of x_{ij} to be s_{ij} and extend by multiplicativity. We will write s_P for the \mathfrak{S}_n -degree of an \mathfrak{S}_n -homogeneous element $P \in \mathcal{E}_n$. We will use the word “homogeneous” to mean homogeneous with respect to both the usual grading and the \mathfrak{S}_n -grading.

The Fomin-Kirillov algebra also has an \mathfrak{S}_n -action given by permuting the indices of the generators x_{ij} . In other words, for $w \in \mathfrak{S}_n$, $w(x_{ij}) = x_{w(i)w(j)}$, and we extend by multiplicativity. This induces an automorphism of \mathcal{E}_n .

Using this gradation and action, \mathcal{E}_n can be thought of as an object in a braided monoidal category (or more specifically, in the *Yetter-Drinfeld category* over $\mathbf{C}[S_n]$). In other words, we define a braiding $\tau: \mathcal{E}_n \otimes \mathcal{E}_n \rightarrow \mathcal{E}_n \otimes \mathcal{E}_n$ by $\tau(P \otimes Q) = Q \otimes s_Q^{-1}(P)$ for homogeneous $P, Q \in \mathcal{E}_n$. We use this to define a braided product structure on $\mathcal{E}_n \otimes \mathcal{E}_n$ via

$$(P_1 \otimes P_2)(Q_1 \otimes Q_2) = P_1 Q_1 \otimes s_{Q_1}^{-1}(P_2) Q_2.$$

Remark 2.5. This convention for the braiding is different from the usual convention, but we use it here for ease of compatibility with the definitions of skew divided difference operators.

2.3.2. Coproduct. There exists a coproduct Δ on \mathcal{E}_n defined by $x_{ij} \mapsto x_{ij} \otimes 1 + 1 \otimes x_{ij}$ and extended to all of \mathcal{E}_n as a braided homomorphism.

Example 2.6. We can compute $\Delta(x_{12}x_{23})$:

$$\begin{aligned} \Delta(x_{12}x_{23}) &= (x_{12} \otimes 1 + 1 \otimes x_{12})(x_{23} \otimes 1 + 1 \otimes x_{23}) \\ &= (x_{12} \otimes 1)(x_{23} \otimes 1) + (x_{12} \otimes 1)(1 \otimes x_{23}) + (1 \otimes x_{12})(x_{23} \otimes 1) + (1 \otimes x_{12})(1 \otimes x_{23}) \\ &= x_{12}x_{23} \otimes 1 + x_{12} \otimes x_{23} + x_{23} \otimes x_{12} + 1 \otimes x_{12}x_{23}. \end{aligned}$$

Compare this calculation to Example 2.2.

The main reason for introducing the coproduct structure is the following proposition.

Proposition 2.7. *For any $w \in \mathfrak{S}_n$, $\Delta(x_w) = \sum_{v \in \mathfrak{S}_n} x_v \otimes x_{w/v}$.*

Proof. Follows from the definition of $x_{w/v}$ and the coproduct structure. \square

Note that if $P \in \mathcal{E}_n$ is a monomial, then any term appearing in the first tensor factor of $\Delta(P)$ is the product of a subsequence of variables in P . (This is not the case for the second tensor factor due to the braiding.)

2.3.3. *Pairing.* There exists a unique linear map $\Delta_{ab}: \mathcal{E}_n \rightarrow \mathcal{E}_n$ satisfying

$$\Delta_{ab}(x_{ij}) = \begin{cases} 1, & \text{if } i = a, j = b; \\ -1, & \text{if } i = b, j = a; \\ 0, & \text{otherwise;} \end{cases}$$

and $\Delta_{ab}(PQ) = \Delta_{ab}(P) \cdot Q + s_{ab}(P) \cdot \Delta_{ab}(Q)$ for all $P, Q \in \mathcal{E}_n$. The operators Δ_{ab} satisfy the relations of \mathcal{E}_n , so they define a left action of \mathcal{E}_n on itself. We can think of Δ_{ab} as having degree -1 and \mathfrak{S}_n -degree s_{ab} . For any $P \in \mathcal{E}_n$, we will write Δ_P for the corresponding operator; in other words, if $P = x_{i_1 j_1} \cdots x_{i_k j_k}$, then $\Delta_P = \Delta_{i_1 j_1} \cdots \Delta_{i_k j_k}$, and we extend by linearity.

There is likewise a dual action: one can define $\nabla_{ab}: \mathcal{E}_n \rightarrow \mathcal{E}_n$ (acting on the right) satisfying $(x_{ij})\nabla_{ab} = \Delta_{ab}(x_{ij})$ and $(PQ)\nabla_{ab} = P \cdot (Q)\nabla_{ab} + (P)(s_Q \nabla_{ab}) \cdot Q$, where $s_Q \nabla_{ab} = \nabla_{s_Q(a)s_Q(b)}$ for homogeneous $P, Q \in \mathcal{E}_n$. Then the operators ∇_{ab} define a right action of \mathcal{E}_n on itself. We define ∇_P for any $P \in \mathcal{E}_n$ as above.

Note that when ∇_{ab} acts on a monomial P , it results in a linear combination of monomials obtained from P by removing a variable. In other words, $(P)\nabla_{ab}$, and in general $(P)\nabla_Q$, has an expression that only contains variables appearing in P . (This is not the case for the action of Δ_{ab} due to the twisting action.)

Example 2.8. Here are two computations involving these operators:

$$\begin{aligned} \Delta_{23}(x_{12}x_{23}x_{12}) &= \Delta_{23}(x_{12}) \cdot x_{23}x_{12} + x_{13} \cdot \Delta_{23}(x_{23}) \cdot x_{12} + x_{13}x_{32} \cdot \Delta_{23}(x_{12}) \\ &= x_{13}x_{12}, \\ (x_{12}x_{23}x_{12})\nabla_{23} &= x_{12}x_{23} \cdot (x_{12})\nabla_{23} + x_{12} \cdot (x_{23})\nabla_{13} \cdot x_{12} + (x_{12})\nabla_{12} \cdot x_{23}x_{12} \\ &= x_{23}x_{12}. \end{aligned}$$

If P and Q are homogeneous of the same degree, then $\Delta_P(Q) = \Delta_Q(P) = (P)\nabla_Q = (Q)\nabla_P$. If we write $\langle P, Q \rangle = \Delta_P(Q)$, then this defines a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathcal{E}_n . With respect to this form, Δ_P and ∇_P are adjoint to right and left multiplication by P , respectively. If P and Q are homogeneous, then $\langle P, Q \rangle = 0$ unless $\deg P = \deg Q$ and $s_P = s_Q^{-1}$.

An alternative way to describe Δ_P and ∇_P is in terms of this bilinear form and the coproduct. If $\Delta(Q) = \sum Q_{(1)}^i \otimes Q_{(2)}^i$, then

$$\begin{aligned} \Delta_P(Q) &= \sum \langle P, Q_{(1)}^i \rangle \cdot Q_{(2)}^i, \\ (Q)\nabla_P &= \sum Q_{(1)}^i \cdot \langle Q_{(2)}^i, P \rangle. \end{aligned}$$

Note that by the cocommutativity of the coproduct, Δ_{P_1} and ∇_{P_2} commute: if $(\Delta \otimes \Delta)(Q) = \sum Q_{(1)}^i \otimes Q_{(2)}^i \otimes Q_{(3)}^i$, then

$$\Delta_{P_1}(Q) \nabla_{P_2} = \sum \langle P_1, Q_{(1)}^i \rangle \cdot Q_{(2)}^i \cdot \langle Q_{(3)}^i, P_2 \rangle.$$

2.3.4. *Antipode.* The antipode $S: \mathcal{E}_n \rightarrow \mathcal{E}_n$ is defined by $x_{ij} \mapsto -x_{ij}$, extended to all of \mathcal{E}_n as a braided antihomomorphism. In other words, if $\mu: \mathcal{E}_n \otimes \mathcal{E}_n \rightarrow \mathcal{E}_n$ is the multiplication map, then

$$S(PQ) = S(\mu(P \otimes Q)) = \mu(\tau(S(P) \otimes S(Q))) = \mu(S(Q) \otimes s_Q^{-1}(S(P))) = S(Q) \cdot s_Q^{-1}(S(P)).$$

The antipode preserves \mathfrak{S}_n -degree.

Example 2.9. The antipode of $x_{12}x_{23}x_{34}$ is

$$\begin{aligned} S(x_{12}x_{23}x_{34}) &= S(x_{34}) \cdot s_{34}(S(x_{12}x_{23})) \\ &= S(x_{34}) \cdot s_{34}(S(x_{23})) \cdot s_{34}s_{23}(S(x_{12})) \\ &= (-x_{34}) \cdot (-x_{24}) \cdot (-x_{14}) \\ &= -x_{34}x_{24}x_{14}. \end{aligned}$$

It will be helpful for us to introduce a variant of the antipode. Let $\rho: \mathcal{E}_n \rightarrow \mathcal{E}_n$ be the map that reverses the order of any monomial, and for any homogeneous P , let $\overline{S}(P) = (-1)^{\deg P} \rho(S(P))$. For example, $\overline{S}(x_{12}x_{23}x_{34}) = x_{14}x_{24}x_{34}$. Note that $s_{\overline{S}(P)} = s_P^{-1}$.

From this definition, it is easy to check that $\overline{S}(x_{i_1 j_1} \cdots x_{i_\ell j_\ell}) = y_1 \cdots y_\ell$, where $y_k = s_{i_\ell j_\ell} \cdots s_{i_{k+1} j_{k+1}}(x_{i_k j_k})$.

The following proposition gives some important properties of \overline{S} .

Proposition 2.10. (a) For homogeneous $P, Q \in \mathcal{E}_n$, $\overline{S}(PQ) = s_Q^{-1}(\overline{S}(P)) \cdot \overline{S}(Q)$.

(b) The map \overline{S} is an involution.

(c) Let $\overline{\tau}: \mathcal{E}_n \otimes \mathcal{E}_n \rightarrow \mathcal{E}_n \otimes \mathcal{E}_n$ be the linear map that switches the two tensor factors (without twisting). Then $\Delta \circ \overline{S} = \overline{\tau} \circ (\overline{S} \otimes \overline{S}) \circ \Delta$.

(d) For any $P \in \mathcal{E}_n$, $\Delta_{ab}(\overline{S}(P)) = \overline{S}((P) \nabla_{ab})$.

(e) The operators \overline{S} and ρ are adjoint with respect to $\langle \cdot, \cdot \rangle$.

Proof. For (a),

$$\begin{aligned} \overline{S}(PQ) &= (-1)^{\deg PQ} \cdot \rho(S(PQ)) \\ &= (-1)^{\deg P + \deg Q} \cdot \rho(S(Q) \cdot s_Q^{-1}(S(P))) \\ &= (-1)^{\deg P} s_Q^{-1}(\rho(S(P))) \cdot (-1)^{\deg Q} \rho(S(Q)) \\ &= s_Q^{-1}(\overline{S}(P)) \cdot \overline{S}(Q). \end{aligned}$$

For (b), we induct on degree. It is clear that \overline{S}^2 is the identity in degree 0 and 1. Then for higher degrees,

$$\overline{S}^2(PQ) = \overline{S}(s_Q^{-1}(\overline{S}(P)) \cdot \overline{S}(Q)) = s_{\overline{S}(Q)}^{-1} \overline{S}(s_Q^{-1}(\overline{S}(P))) \cdot \overline{S}^2(Q) = s_Q s_Q^{-1}(\overline{S}^2(P)) \cdot Q = PQ.$$

For (c), we again induct on degree. Again the claim is clear in degree 0 or 1. Suppose $\Delta(P) = \sum P_{(1)}^i \otimes P_{(2)}^i$ and $\Delta(Q) = \sum Q_{(1)}^j \otimes Q_{(2)}^j$. Then by induction,

$$\begin{aligned}
\Delta \circ \bar{S}(PQ) &= \Delta(s_Q^{-1}(\bar{S}(P)) \cdot \bar{S}(Q)) \\
&= s_Q^{-1} \Delta(\bar{S}(P)) \cdot \Delta(\bar{S}(Q)) \\
&= s_Q^{-1}(\bar{\tau} \circ (\bar{S} \otimes \bar{S}) \circ \Delta(P)) \cdot \bar{\tau} \circ (\bar{S} \otimes \bar{S}) \circ \Delta(Q) \\
&= \sum_{i,j} s_Q^{-1}(\bar{S}(P_{(2)}^i) \otimes \bar{S}(P_{(1)}^i)) \cdot (\bar{S}(Q_{(2)}^j) \otimes \bar{S}(Q_{(1)}^j)) \\
&= \sum_{i,j} s_Q^{-1}(\bar{S}(P_{(2)}^i)) \bar{S}(Q_{(2)}^j) \otimes s_{\bar{S}(Q_{(2)}^j)}^{-1}(s_Q^{-1}(\bar{S}(P_{(1)}^i))) \bar{S}(Q_{(1)}^j) \\
&= \sum_{i,j} s_{Q_{(2)}^j}^{-1} \bar{S}(s_{Q_{(1)}^i}^{-1}(P_{(2)}^i)) \bar{S}(Q_{(2)}^j) \otimes s_{Q_{(1)}^i}^{-1}(\bar{S}(P_{(1)}^i)) \bar{S}(Q_{(1)}^j) \\
&= \sum_{i,j} \bar{S}(s_{Q_{(1)}^i}^{-1}(P_{(2)}^i) Q_{(2)}^j) \otimes \bar{S}(P_{(1)}^i Q_{(1)}^j) \\
&= \sum_{i,j} \bar{\tau} \circ (\bar{S} \otimes \bar{S})(P_{(1)}^i Q_{(1)}^j \otimes s_{Q_{(1)}^i}^{-1} P_{(2)}^i Q_{(2)}^j) \\
&= \bar{\tau} \circ (\bar{S} \otimes \bar{S}) \circ \Delta(PQ).
\end{aligned}$$

For (d),

$$\begin{aligned}
\Delta \circ \bar{S}(P) &= \bar{\tau} \circ (\bar{S} \otimes \bar{S}) \circ \Delta(P) \\
&= \sum_i \bar{\tau} \circ (\bar{S} \otimes \bar{S})(P_{(1)}^i \otimes P_{(2)}^i) \\
&= \sum_i \bar{S}(P_{(2)}^i) \otimes \bar{S}(P_{(1)}^i).
\end{aligned}$$

Thus

$$\Delta_{ab}(\bar{S}(P)) = \sum_i \langle x_{ab}, \bar{S}(P_{(2)}^i) \rangle \cdot \bar{S}(P_{(1)}^i) = \sum_i \langle x_{ab}, P_{(2)}^i \rangle \cdot \bar{S}(P_{(1)}^i) = \bar{S}((P) \nabla_{ab}).$$

For (e), if P and Q are homogeneous of the same degree, then

$$\langle Q, \bar{S}(P) \rangle = \Delta_Q(\bar{S}(P)) = \bar{S}((P) \Delta_{\rho(Q)}) = (P) \Delta_{\rho(Q)} = \langle P, \rho(Q) \rangle. \quad \square$$

Remark 2.11. Propostion 2.10(e) easily implies that the antipode S is self-adjoint, but we will not need this result below.

Since we did not use any of the relations of \mathcal{E}_n in proving Proposition 2.10, the analogous result holds even in the full tensor algebra with no relations. In particular, for part (c), if P is a monomial of degree d , then the 2^d terms obtained from expanding $\Delta \circ \bar{S}(P)$ are identical to the 2^d terms obtained from expanding $\bar{\tau} \circ (\bar{S} \otimes \bar{S}) \circ \Delta(P)$ without using any relations between the x_{ij} (even $x_{ij} = -x_{ji}$).

In the next section, we will use the properties presented above to prove Theorem 2.4.

3. POSITIVITY

Let us denote by \mathcal{E}_n^+ the positive span of monomials in variables x_{ij} with $i < j$. In this section, we will prove that $x_{w/v} \in \mathcal{E}_n^+$ as conjectured by Kirillov [9]. This will immediately imply Theorem 2.4.

First we relate the operators and pairing described in the previous section to the Bruhat order.

Proposition 3.1. (a) Let $w \in \mathfrak{S}_n$, and choose any $x_{ij} \in \mathcal{E}_n^+$. Then $(x_w)\nabla_{ij} = x_{ws_{ij}}$ if $ws_{ij} \triangleleft w$ and 0 otherwise.
(b) Let $v, w \in \mathfrak{S}_n$. Then $(x_w)\nabla_v = x_{v'}$ if there exists a reduced factorization $w = v' \cdot v^{-1}$ and 0 otherwise.

Proof. Let $w = s_{a_1} \cdots s_{a_\ell}$. From the definition of coproduct, the degree $(\ell - 1, 1)$ part of $\Delta(x_w)$ is

$$\sum_{k=1}^{\ell} x_{a_1, a_1+1} \cdots \widehat{x}_{a_k, a_k+1} \cdots x_{a_\ell, a_\ell+1} \otimes s_{a_\ell} \cdots s_{a_{k+1}}(x_{a_k, a_k+1}).$$

The first tensor factor is nonzero if and only if $s_{a_1} \cdots \widehat{s}_{a_k} \cdots s_{a_\ell}$ is a reduced expression for some $v \in \mathfrak{S}_n$ with $v \triangleleft w$, and in this case, the second tensor factor must be, up to sign, $x_{v^{-1}w}$.

If $v \triangleleft w$, then by the strong exchange condition for Bruhat order, there is a unique k such that $s_{a_1} \cdots \widehat{s}_{a_k} \cdots s_{a_\ell}$ is a reduced expression for v . Hence $(x_w)\nabla_{ij} = 0$ unless $v = ws_{ij} \triangleleft w$, in which case $(x_w)\nabla_{ij} = x_v \cdot \langle s_{a_\ell} \cdots s_{a_{k+1}}(x_{a_k, a_k+1}), x_{ij} \rangle = \pm x_v$. But in fact this must have a positive sign: since $s_{a_k} \cdot s_{a_{k+1}} \cdots s_{a_\ell} > s_{a_{k+1}} \cdots s_{a_\ell}$, it follows that $s_{a_\ell} \cdots s_{a_{k+1}}(a_k) < s_{a_\ell} \cdots s_{a_{k+1}}(a_k + 1)$, so the former must be i and the latter j (since $i < j$ by assumption).

For part (b), choose any reduced expression for v and apply part (a) repeatedly. \square

By iterating this result, we can prove the following consequence.

Proposition 3.2. Let $w \in \mathfrak{S}_n$, and let $P = x_{i_1 j_1} \cdots x_{i_\ell j_\ell} \in \mathcal{E}_n^+$ be any monomial. Let $v_k = s_{i_k j_k} s_{i_{k+1} j_{k+1}} \cdots s_{i_\ell j_\ell}$. Then $\langle x_w, P \rangle = 1$ if

$$\text{id} \triangleleft v_\ell \triangleleft v_{\ell-1} \triangleleft \cdots \triangleleft v_1 = w^{-1}$$

is a saturated chain in the Bruhat order of \mathfrak{S}_n ; otherwise $\langle x_w, P \rangle = 0$.

In particular, for $v, w \in \mathfrak{S}_n$, $\langle x_v, x_w \rangle = 1$ if $w = v^{-1}$ and 0 otherwise.

Proof. Clearly for $\langle x_w, P \rangle$ to be nonzero, we must have $\ell = \ell(w) = \deg(P)$. We proceed by induction on ℓ .

Note $\langle x_w, P \rangle = \langle (x_w)\nabla_{i_1 j_1}, P' \rangle$, where $P = x_{i_1 j_1} P'$. This is nonzero if and only if, by Proposition 3.1, $ws_{i_1 j_1} \triangleleft w$ and, by induction, $\text{id} \triangleleft v_\ell \triangleleft v_{\ell-1} \triangleleft \cdots \triangleleft v_2 = (ws_{i_1 j_1})^{-1}$ is a saturated Bruhat chain, in which case $\langle x_w, P \rangle = 1$. But then

$$v_2 = s_{i_1 j_1} w^{-1} \triangleleft w^{-1} = s_{i_1 j_1} v_2 = v_1,$$

completing the Bruhat chain, as desired. \square

We will also need the following result about $\overline{S}(x_w)$.

Proposition 3.3. Let $w \in \mathfrak{S}_n$. Then $\overline{S}(x_w) \in \mathcal{E}_n^+$. In fact, the variables appearing in $\overline{S}(x_w)$ are precisely those x_{ij} for which $i < j$ and $w(i) > w(j)$.

Proof. Let $w = s_{i_1} \dots s_{i_\ell}$ be a reduced expression, and let $w_k = s_{i_k} \dots s_{i_\ell}$. Then by definition, $\overline{S}(x_w) = y_1 \dots y_\ell$, where $y_k = w_{k+1}^{-1}(x_{i_k, i_k+1})$. Since $w_{k+1}^{-1}s_{i_k} = w_k^{-1} \succ w_{k+1}^{-1}$, we must have $a = w_{k+1}^{-1}(i_k) < w_{k+1}^{-1}(i_k + 1) = b$, so $y_k = x_{ab} \in \mathcal{E}_n^+$. Then $w_k(a) = i_k + 1 > i_k = w_k(b)$. Since multiplying on the left by simple transpositions cannot remove inversions without decreasing length, it follows that $w(a) > w(b)$, as desired. \square

Proposition 3.3 implies that, in particular, $\overline{S}(x_{w_0}) \in \mathcal{E}_n^+$. In fact, we can say more about $\overline{S}(x_{w_0})$. This will be related to the following definition from the theory of Coxeter groups (see [2, 4]).

Definition. A *reflection ordering* for the transpositions $s_{ij} \in \mathfrak{S}_n$ is a total order \prec such that for any $i < j < k$, s_{ik} lies somewhere between s_{ij} and s_{jk} .

There is an equivalent formulation: $t_1 \prec \dots \prec t_N$ is a reflection ordering if and only if there exists a reduced expression $w_0 = s_{i_1} \dots s_{i_N}$ such that $t_k = s_{i_N} \dots s_{i_{k+1}} s_{i_k} s_{i_{k+1}} \dots s_{i_N}$. In other words, if $s_{i_1 j_1} \prec \dots \prec s_{i_N j_N}$ is a reflection ordering, then $x_{i_1 j_1} \dots x_{i_N j_N} = \overline{S}(x_{w_0})$. We are now ready to prove the following proposition.

Proposition 3.4. In \mathcal{E}_n , $x_{w_0} = \overline{S}(x_{w_0}) = x_{i_1 j_1} \dots x_{i_N j_N} \in \mathcal{E}_n^+$, where $s_{i_1 j_1} \prec \dots \prec s_{i_N j_N}$ is any reflection ordering.

Proof. It suffices to prove the claim for a fixed reflection ordering, say

$$s_{12} \prec s_{13} \prec s_{23} \prec s_{14} \prec s_{24} \prec s_{34} \prec \dots \prec s_{1n} \prec \dots \prec s_{n-1,n}.$$

We induct on n . Let w'_0 be the longest element of S_{n-1} , so $w = w'_0 \cdot s_{n-1} s_{n-2} \dots s_1$ is a reduced factorization. By the inductive hypothesis, $x_{w'_0} = x_{12} x_{13} x_{23} \dots x_{n-2, n-1} \in \mathcal{E}_{n-1}$. Since w'_0 has maximum length in S_{n-1} , $x_{w'_0} x_{i, i+1} = 0$ for $i = 1, \dots, n-2$. Then

$$\begin{aligned} x_{w'_0} \cdot x_{1n} x_{2n} x_{3n} \dots x_{n-1, n} &= x_{w'_0} \cdot (-x_{12} x_{1n} + x_{2n} x_{12}) \cdot x_{3n} \dots x_{n-1, n} \\ &= x_{w'_0} \cdot x_{2n} x_{12} \cdot x_{3n} \dots x_{n-1, n} \\ &= x_{w'_0} \cdot x_{2n} x_{3n} \dots x_{n-1, n} \cdot x_{12} \\ &= x_{w'_0} \cdot (-x_{23} x_{2n} + x_{3n} x_{23}) \dots x_{n-1, n} \cdot x_{12} \\ &= x_{w'_0} \cdot x_{3n} \dots x_{n-1, n} \cdot x_{23} x_{12} \\ &= \dots \\ &= x_{w'_0} \cdot x_{n-1, n} x_{n-2, n-1} \dots x_{23} x_{12} \\ &= x_{w_0}. \end{aligned} \quad \square$$

We are now ready to prove our main theorem.

Theorem 3.5. For any $v, w \in \mathfrak{S}_n$, $x_{w/v} \in \mathcal{E}_n^+$.

Proof. Since $\ell(w_0) = \ell(w) + \ell((w_0 w)^{-1})$, by Proposition 3.1(b), $(x_{w_0}) \nabla_{w_0 w} = x_w$. Then by Propositions 2.7 and 3.2, $\Delta_{v^{-1}}(x_w) = \sum_{v'} \langle x_{v^{-1}}, x_{v'} \rangle \cdot x_{w/v'} = x_{w/v}$. Thus

$$x_{w/v} = \Delta_{v^{-1}}(x_{w_0}) \nabla_{w_0 w}.$$

Now by Propositions 3.4, 2.10(d), and 3.1(b),

$$\Delta_{v^{-1}}(x_{w_0}) = \Delta_{v^{-1}}(\overline{S}(x_{w_0})) = \overline{S}((x_{w_0}) \nabla_v) = \overline{S}(x_{w_0 v}).$$

Write $\Delta(\overline{S}(x_{w_0v})) = \sum X_{(1)}^i \otimes X_{(2)}^i$. Then since $\overline{S}(x_{w_0v}) \in \mathcal{E}_n^+$ by Proposition 3.3, we also have $X_{(1)}^i \in \mathcal{E}_n^+$. Moreover, by Propositions 2.10(c) and 2.7,

$$\sum X_{(2)}^i \otimes X_{(1)}^i = \overline{\tau} \circ \Delta \circ \overline{S}(x_{w_0v}) = (\overline{S} \otimes \overline{S}) \circ \Delta(x_{w_0v}) = \sum_u \overline{S}(x_u) \otimes \overline{S}(x_{w_0v/u}).$$

Hence if $X_{(2)}^i$ is nonzero, then it equals $\overline{S}(x_u)$ for some $u \in \mathfrak{S}_n$ and therefore lies in \mathcal{E}_n^+ . Then

$$x_{w/v} = (\overline{S}(x_{w_0v})) \nabla_{w_0w} = \sum X_{(1)}^i \cdot \langle X_{(2)}^i, x_{w_0w} \rangle.$$

Since $\langle X_{(2)}^i, x_{w_0w} \rangle$ is either 0 or 1 by Proposition 3.2, $x_{w/v}$ lies in \mathcal{E}_n^+ , as desired. \square

Tracing through the proof of Theorem 3.5, we can write down an explicit expression for $x_{w/v}$.

Corollary 3.6. *Let $v, w \in \mathfrak{S}_n$. Choose any reduced expression $w_0v = s_{i_1} \cdots s_{i_\ell}$, and let $w_k = s_{i_k} \cdots s_{i_\ell}$. Define $y_k = w_{k+1}^{-1}(x_{i_k, i_k+1}) \in \mathcal{E}_n^+$. Then*

$$x_{w/v} = \sum_J \prod_{k \notin J} y_k,$$

where $J \subseteq \{1, \dots, \ell\}$ ranges over all subsets such that $\prod_{k \in J} s_{i_k}$ is a reduced expression for w_0w .

Proof. As in Theorem 3.5, $x_{w/v} = (\overline{S}(x_{w_0v})) \nabla_{w_0w} = (y_1 \cdots y_\ell) \nabla_{w_0w}$. By the proof of Theorem 3.5,

$$\Delta(y_1 \cdots y_\ell) = \sum_{J \subseteq \{1, \dots, \ell\}} ((\prod_{k \notin J} y_k) \otimes \overline{S}(\prod_{k \in J} x_{i_k, i_k+1})).$$

By Proposition 2.10(e) $\langle \overline{S}(x_u), x_{w_0w} \rangle = \langle x_u, x_{w^{-1}w_0} \rangle$, which by Proposition 3.2 equals 1 if $u = w_0w$ and 0 otherwise. The result follows. \square

Example 3.7. Let $w = s_2s_1s_3s_2 \in \mathfrak{S}_4$ and $v = s_2$. One reduced expression for w_0v is $s_3s_2s_1s_2s_3$. Then $\overline{S}(x_{w_0v}) = x_{13}x_{12}x_{14}x_{24}x_{34}$. Since $w_0w = s_1s_3 = s_3s_1$, there are two reduced subexpressions for w_0w in w_0v , namely for $J = \{1, 3\}$ and $J = \{3, 5\}$. Removing the corresponding variables from $\overline{S}(x_{w_0v})$ gives

$$x_{w/v} = x_{12}x_{24}x_{34} + x_{13}x_{12}x_{24}.$$

Compare this calculation to Example 2.3.

Each term in the expansion of $x_{w/v}$ corresponds to a reduced subword for w_0w lying inside a reduced expression for w_0v . For more information about these subwords, see [10].

Interestingly, Corollary 3.6 implies that $x_{w/v}$ has an expression in which no variable is repeated in any monomial.

One can also use Theorem 3.5 to give a positive recurrence for $x_{w/v}$.

Corollary 3.8. *Let $v, w \in \mathfrak{S}_n$, and suppose $a = v^{-1}(i) < v^{-1}(i+1) = b$. Let $v' = s_i v$ and $w' = s_i w$. If $w \leq w'$, then $x_{w/v} = x_{ab}x_{w/v'} + x_{w'/v'}$; otherwise $x_{w/v} = x_{ab}x_{w/v'}$.*

Proof. If $v^{-1}(i) < v^{-1}(i+1)$, then $(w_0v)^{-1}(n+1-i) < (w_0v)^{-1}(n-i)$. Hence there exists a reduced expression for w_0v that starts with s_{n-i} . Let us fix such a reduced expression and apply Corollary 3.6. In this case $y_1 = x_{ab}$, and $y_2 \cdots y_\ell = \overline{S}(x_{s_{n-i}w_0v}) = \overline{S}(x_{w_0v'})$.

Suppose $J \subseteq \{1, \dots, \ell\}$ gives a reduced subexpression for w_0w . If $1 \notin J$, then J gives a reduced subexpression for w_0w in a reduced expression for w_0v' . This then contributes $x_{ab}x_{w/v'}$ to $x_{w/v}$. If $1 \in J$, then we must have $w_0w' = s_{n-i}w_0w \prec w_0w$, or equivalently $w \prec w'$, and $J \setminus \{1\}$ must give a reduced subexpression for w_0w' in a reduced expression for w_0v' . Hence this contributes $x_{w'/v'}$ to $x_{w/v}$ (provided $w \prec w'$). The result follows. \square

Corollary 3.8 then gives an explicit positive recurrence that implies $x_{w/v} \in \mathcal{E}_n^+$ (noting that if $\ell(v) = \ell(w)$, then $x_{w/v} = 1$ if $v = w$ and 0 otherwise).

REFERENCES

- [1] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. Pointed Hopf algebras. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 1–68. Cambridge Univ. Press, Cambridge, 2002.
- [2] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.
- [3] Jonah Blasiak, Ricky Ini Liu, and Karola Mészáros. Subalgebras of the Fomin-Kirillov algebra. *Preprint*, 2012. [arxiv:1310.4112](#).
- [4] Francesco Brenti, Sergey Fomin, and Alexander Postnikov. Mixed Bruhat operators and Yang-Baxter equations for Weyl groups. *Internat. Math. Res. Notices*, (8):419–441, 1999.
- [5] Sergey Fomin and Anatol N. Kirillov. Quadratic algebras, Dunkl elements, and Schubert calculus. In *Advances in geometry*, volume 172 of *Progr. Math.*, pages 147–182. Birkhäuser Boston, Boston, MA, 1999.
- [6] Sergey Fomin and Claudio Procesi. Fibered quadratic Hopf algebras related to Schubert calculus. *J. Algebra*, 230(1):174–183, 2000.
- [7] Sergey Fomin and Richard P. Stanley. Schubert polynomials and the nil-Coxeter algebra. *Adv. Math.*, 103(2):196–207, 1994.
- [8] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
- [9] Anatol N. Kirillov. Skew divided difference operators and Schubert polynomials. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 3:Paper 072, 14, 2007.
- [10] Allen Knutson and Ezra Miller. Subword complexes in Coxeter groups. *Adv. Math.*, 184(1):161–176, 2004.
- [11] I. G. Macdonald. Schubert polynomials. In *Surveys in combinatorics, 1991 (Guildford, 1991)*, volume 166 of *London Math. Soc. Lecture Note Ser.*, pages 73–99. Cambridge Univ. Press, Cambridge, 1991.
- [12] Alexander Milinski and Hans-Jürgen Schneider. Pointed indecomposable Hopf algebras over Coxeter groups. In *New trends in Hopf algebra theory (La Falda, 1999)*, volume 267 of *Contemp. Math.*, pages 215–236. Amer. Math. Soc., Providence, RI, 2000.

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